Assignment 8

1. Using Heun's method, approximate $y(1)$ with $h = 0.2$ and again with $h = 0.1$ for the initial-value problem defined by

$$y^{(1)}(t) = 2y(t) + t - 1$$
$$y(0) = 1$$

2. In Question 1, you approximated $y(0.2)$ with $h = 0.2$, and $y(0.1)$ with $h = 0.1$. The correct solutions to sixteen significant digits are $y(0.2) = 1.268868523230952$ and $y(0.1) = 1.116052068620128$. Show that the error of one step of Heun's method is $O(h^3)$ by showing that the error of your approximation at $t = 0.1$ is approximately one eighth the error at $t = 0.2$. You should consider doing one step by hand, and then multiple steps using a program.

3. Using the 4th-order Runge-Kutta method, approximate $y(1)$ with $h = 0.2$ and again with $h = 0.1$ for the initial-value problem defined by

$$y^{(1)}(t) = 2y(t) + t - 1$$
$$y(0) = 1$$

4. In Question 1, you approximated $y(0.2)$ with $h = 0.2$, and $y(0.1)$ with $h = 0.1$. The correct solutions to sixteen significant digits are $y(0.2) = 1.268868523230952$ and $y(0.1) = 1.116052068620128$. Show that the error of one step of 4th-order Runge-Kutta method is $O(h^5)$ by showing that the error of your approximation at $t = 0.1$ is approximately one thirty-second the error at $t = 0.2$. You should consider doing one step by hand, and then multiple steps using a program.

5. When we do the error analysis on Euler's method, are we using a convex combination of the errors, or a more general weighted average of the errors? Thus, may we say that the error is bounded above and below by the second derivative evaluated at some point on the interval, or may this not be the case?

6. For argument's sake, how small would h have to be so that the error is less than 0.01 using Euler's method to find the approximation to the initial-value problem

$$y^{(1)}(t) = -y(t)$$
$$y(0) = 1$$

to approximate the value of $y(10)$. You can find h because you know the solution to this initial-value problem and you can calculate an upper bound for the solution's second derivative on the interval [0, 10].

7. Apply one step of our adaptive Euler-Heun to approximate a solution to the initial-value problem

$$y^{(1)}(t) = 2y(t) + t - 1$$
$$y(0) = 1$$

starting with an $h = 0.1$ with the per unit time error $e_{\text{abs}} = 0.1$. What value of h would you use with the next step, and would you be recalculating the previous step, or calculating the next step?

8. Suppose you were applying the Dormand-Prince method and you started with $h = 0.1$ with a per unit time acceptable error of $\varepsilon_{abs} = 0.00001$, and your two approximations of the next point were

$$y = 1.1160522588 \text{ and } z = 1.11605208.$$

What would your value of a be in this case, and what step size would you use with the next step? Would you be recalculating this point with the same h value, or would you continue to approximate the next point?

9. Incidentally, the approximations y and z are the approximations to the solution to the initial-value problem given in Question 7 approximating the solution at $y(0.1)$. Given that the exact solution to sixteen significant digits is $y(0.1) = 1.116052068620128$, demonstrate that the error introduced is indeed less than $h\varepsilon_{abs}$.